Pf $\quad \nabla_X Y = \frac{1}{2}\Big([X,Y] + \underbrace{\cancel{(\mathrm{ad}_X)(Y)} + \cancel{(\mathrm{ad}_Y)(X)}}_{\text{cancelled}}\Big) = \frac{1}{2}[X,Y]. \quad \square$

Rmk In particular, $X=Y \Rightarrow \nabla_X X = \frac{1}{2}[X,X] = 0.$ $\leftarrow$ therefore, the integral curve of a left inv v.f. under bi-inv metric g on G is a geodesic [illegible].

How about curvature ?

From (∇), observe that if X, Y are left inv. v.f. then $\nabla_X Y$ is also a left inv. v.f. Therefore

$$Y\, g(\nabla_X Z, W) = 0 \quad \text{for all left inv input v.fs.}$$

$$\Rightarrow \quad 0 = Y\, g(\nabla_X Z, W)$$

defining axiom $\longrightarrow$

$$= g(\nabla_Y \nabla_X Z, W) + g(\nabla_X Z, \nabla_Y W)$$

$$\Rightarrow \quad g(\nabla_Y \nabla_X Z, W) = -g(\nabla_X Z, \nabla_Y W).$$

Therefore

$$R(X,Y,Z,W) \overset{\text{def}}{=} g(R(X,Y)Z,W)$$

$$\overset{\text{def}}{=} g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W)$$

$$= -g(\nabla_Y Z, \nabla_X W) + g(\nabla_X Z, \nabla_Y W) - g(\nabla_{[X,Y]}Z, W)$$

Now, assume g is bi-inv, then recall prop above shows $\nabla_X Y = \frac{1}{2}[X,Y]$.

Then

$$R(X,Y,Z,W) = -g\left(\tfrac{1}{2}[Y,Z], \tfrac{1}{2}[X,W]\right) + g\left(\tfrac{1}{2}[X,Z], \tfrac{1}{2}[Y,W]\right)$$

$$- g\left(\tfrac{1}{2}[[X,Y],Z], W\right)$$

Jacobi identity

$$= -\tfrac{1}{4} g([Y,Z],[X,W]) + \tfrac{1}{4} g([X,Z],[Y,W])$$

$[[X,Y],Z]$
$= -[[Y,Z],X] - [[Z,X],Y]$

$$+ \tfrac{1}{2} g([[Y,Z],X],W) + \tfrac{1}{2} g([[Z,X],Y],W)$$

$$g([[Y,Z],X],W) = g(-[X,[Y,Z]],W) = g(-\mathrm{ad}_X([Y,Z]),W)$$

$$= g\left((ad_X)^*([Y,Z]), W\right)$$
$$= g\left([Y,Z], (ad_X)(W)\right)$$
$$= g\left([Y,Z], [X,W]\right)$$

$$\Rightarrow R(X,Y,Z,W) = -\frac{1}{4} g([Y,Z],[X,W]) + \frac{1}{4} g([X,Z],[Y,W])$$
$$+ \frac{1}{2} g\left([Y,Z],[X,W]\right) + \frac{1}{2} g\left([Z,X],[Y,W]\right)$$
$$= \frac{1}{4} g\left([Y,Z],[X,W]\right) + \frac{1}{4} g\left([Z,X],[Y,W]\right)$$

In particular,

$$\underset{\substack{\| \quad \| \\ Z \quad W}}{R(X,Y,Y,X)} = \frac{1}{4} g\big(\overset{0}{\cancel{[Y,Y]}}, \overset{0}{\cancel{[X,X]}}\big) + \frac{1}{4} g\left([Y,X],[Y,X]\right)$$
$$= \frac{1}{4} \|[X,Y]\|^2 \geq 0$$

$\Rightarrow$ a rather deep result.

Thm. If a Lie group G admits a bi-inv metric g. Then w.r.t its Levi-Civita connection, the sectional curvature $K(P)$ is always <u>non-negative</u>.

Unfornately, due to time limit of this course, it is impossible to start the principal bundle. We leave this topic to other occasion on next semester – Riem geometry.